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On the Application of Inequalities Containing Sums of Minimum/Maximum of Numbers

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Abstract

Retail inventory management is a crucial part of many businesses due to the high profit associated with it as well as the uncertainty around it, especially for industries with short production cycles and a complex supply chain. Proper management of retail inventories can lead to decreased inventory costs, prevent spoilage and obsolescence, and improve customer satisfaction, all of which lead to increased profits for the company. In this paper, we first stated a well-known inequality. The inequality involves multiple variables and how the maximum/minimum values of a subset of the numbers compare to the maximum/minimum values of the whole set of numbers. After demonstrating this inequality is true, we further proved generalizations of the inequality. With these, we applied our results to the retail inventory management problem. A model for such problem was taken from Ozbay, 2006. Finally, we provided an upper and lower bound for the cost of inventory management. These results might allow companies to make better long term decisions due to the bounds being rigorously proven. However, more work can be done to further improve the bounds in special cases as well as combining the results with other algorithms.

Keywords: Algebraic optimization, Inequalities, Retail inventory management

1. Introduction

Backlogging and inventory holding costs are two major costs of inventory management (Misra, 2022). Such costs arises when there is either extra or shortage of goods. Some notable examples such as the 2017 backlog when Apple launched its iPhone X (Williams, 2017), which caused many to criticise Apple's poor forecasting. Therefore, companies need to calculate the amount of goods that needs to be stored or imported to make sure they always have sufficient goods to meet customer demands.

Inequalities refers to a topic in algebra which investigates the relationship between certain numbers, usually on their size. It often makes use of some common inequalities like AM-GM, Cauchy's inequality (Mitrinovic et al., 2013), or may use other methods like smoothing or even calculus to prove an inequality. Such inequalities are traditionally used in many Mathematics Olympiad questions, with many books covering such area (Sedrakyan et al., 2018; Manfrino et al., 2009; Su et al., 2015). Inequalities that appear in Mathematics contests often comes in the form of proving a certain known identity, and contestants are required to provide a proof showing the above inequality is correct. However, in the past few years, inequalities have seen a wider range of applications from biology (Feng et al., 2016) to managing supplies and inventories in supermarket sales (Agnew, 1975; Moon et al., 2010; C´ardenas-Barr´on et al., 2011).

Retail inventories management is the process of ensuring the company have enough inventory to meet customer demand while that also not having an excess or shortage of goods (Luther, 2021). One classical approach to this problem uses the Economic Order Quantity (EOQ), which is a formula to determine the optimal quantity to order,

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with the amount of ordering given by $\sqrt{2A * O * C}$ (Kumar, 2016; Agarwal, 2014). Another method for inventory management, known as the (s, S) policy, is also widely adopted (Ma et al., 2019). Such problems are sometimes maintained by automated inventory management systems (Samuel, 2012), or may include more complicated methods such as Bayesian methods (DeHoratius et al., 2008) to account for inaccuracies in inventory records.

The goal of this research is to provide explicit algebraic expressions that can bound or estimate the total cost needed for retail inventory management. With these results, companies would be able to estimate the cost for retail inventory management. This would allow them to make better planning, and avoid making the same mistake as Apple. However, due to the complicated nature of such problems, the bounds may not be especially strong and more detailed analysis might be needed to obtain stronger results with more real life applications.

The remainder of this paper is divided as follows. Section 2.1 and 2.2 will first present an inequality and its generalizations. The inequality involves the minimum/maximum of some numbers and the relationship between them. This was then applied to retail inventory management in section 2.3, which used it to strictly bound the cost of inventory storing to determine whether the company has reached the shutdown point in classical economics. The paper finished off with results and limitations (sections 3 and 4) and comparative analysis (section 5).

2. Materials and Methods

2.1 Original Inequality (known)

Most of the following results builds upon an known inequality. The following section provides a prove of that inequality. Before going directly into the proofs, here are some of the notations that are used throughout this paper.

- $min\{a_1, a_2, \cdots, a_n\}$: The minimum of numbers $a_1, a_2, \cdots, a_n$
- $max\{a_1, a_2, \cdots, a_n\}$: The maximum of numbers $a_1, a_2, \cdots, a_n$
- $\forall$: For any
- WLOG: Without Loss of Generality

Besides those notations, the concept of smoothing would also be used later in this paper. Smoothing, at least in the context of inequalities, refers to the idea of making minor adjustments to certain variables. Consider a function which has a minimum when all numbers are equal. To prove that with smoothing, one method is to show that the function is smaller when two unequal values are replaced with two equal ones. This result can then be repeatedly applied to show that the function achieves the minimum when all values are equal. A more detailed explanation can be found online at DanielWainfleet (2019).

Now, moving on to the proofs. First, a short lemma.

Lemma 2.1: $min\,\{a, b\} \le ta + (1-t)b,\ \forall\ 0 \le t \le 1$

Proof. Without loss of generality (WLOG), let $a \le b$

Then, the above is equivalent to $(1-t)a \le (1-t)b$, which immediately follows from $a \le b$ and $1 - t \ge 0$

Next, the following theorem holds:

Theorem 2.2: *Let* $a_1, a_2, \cdots, a_n$ *be real numbers such that* $a_1 + a_2 + \cdots + a_n = 0$.

Then, the following is true

$$\frac{n}{n-1} min\{a_1, a_2, \cdots, a_n\} \ge min\{a_1, a_2\} + min\{a_2, a_3\} + \cdots + min\{a_{n-1}, a_n\} + min\{a_n, a_1\}$$

Proof. WLOG, $a_n = \min\{a_1, a_2, \cdots, a_n\}$

It is then sufficient to show that $-\frac{n-2}{n-1} a_n \ge \min\{a_1, a_2\} + \min\{a_2, a_3\} + \cdots \min\{a_{n-2}, a_{n-1}\}$

Notice that $-a_n = \sum_{i=1}^{n-1} a_i$, Hence, by repeatedly applying lemma 2.1,

$\min\{a_1, a_2\} + \min\{a_2, a_3\} + \cdots \min\{a_{n-2}, a_{n-1}\} \le$

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$$\left(\frac{n-2}{n-1}a_1+\frac{1}{n-1}a_2\right)+\left(\frac{n-3}{n-1}a_2+\frac{2}{n-1}a_3\right)+\cdots\left(\frac{1}{n-1}a_{n-2}+\frac{n-2}{n-1}a_{n-1}\right)=\frac{n-2}{n-1}\sum_{i=1}^{n-1}a_i=-\frac{n-2}{n-1}a_n$$

As desired.

2.2 Generalization of the Inequality

In this section, some generalizations of the above inequality will be presented. First, a corollary.

Corollary 2.3: *Let $a_1, a_2, \cdots, a_n$ be real numbers such that $a_1+a_2+\cdots+a_n=0$.*
Then the following relations holds:

$$\frac{n}{n-1}max\{a_1,a_2,\cdots,a_n\}\le max\{a_1,a_2\}+max\{a_2,a_3\}+\cdots+max\{a_{n-1},a_n\}+max\{a_n,a_1\}$$

Proof. Set $a_i'=-a_i$ and applying Theorem 2.2. Since $\min\{a,b\}=-\max\{-a,-b\}$, $\min\{a_1,a_2,\cdots,a_n\}=-\max\{-a_1,-a_2,\cdots,-a_n\}$ etc, the corollary immediately follows.

The above results can be further generalized.

Theorem 2.4: *Let $a_1, a_2, \cdots, a_n$ be real numbers such that $a_1+a_2+\cdots+a_n=0$. Then*

$$\frac{n}{n-1}min\{a_1,a_2,\cdots,a_n\}\ge min\{a_1,a_2,a_3\}+min\{a_2,a_3,a_4\}+\cdots+min\{a_{n-1},a_n,a_1\}+min\{a_n,a_1,a_2\}$$

$$\frac{n}{n-1}min\{a_1,a_2,\cdots,a_n\}\ge min\{a_1,a_2,a_3,a_4\}+min\{a_2,a_3,a_4,a_5\}+\cdots+min\{a_{n-1},a_n,a_1,a_2\}+min\{a_n,a_1,a_2,a_3\}$$

$$\frac{n}{n-1}min\{a_1,a_2,\cdots,a_n\}\ge min\{a_1,a_2,\cdots,a_{n-1}\}+min\{a_2,a_3,\cdots,a_n\}+\cdots+min\{a_n,a_1,\cdots,a_{n-2}\}$$

Proof. Notice the following inequality: $\min\{a_1,a_2,a_3\}\le\frac{1}{2}\min\{a_1,a_2\}+\frac{1}{2}\min\{a_2,a_3\}$

Hence, $min\{a_1,a_2,a_3\}+min\{a_2,a_3,a_4\}+\cdots+min\{a_{n-1},a_n,a_1\}+min\{a_n,a_1,a_2\}\le$

$$\left(\frac{1}{2}\min\{a_1,a_2\}+\frac{1}{2}\min\{a_2,a_3\}\right)+\left(\frac{1}{2}\min\{a_2,a_3\}+\frac{1}{2}\min\{a_3,a_4\}\right)+\cdots$$

$$+\left(\frac{1}{2}\min\{a_{n-1},a_n\}+\frac{1}{2}\min\{a_n,a_1\}\right)+\left(\frac{1}{2}\min\{a_n,a_1\}+\frac{1}{2}\min\{a_1,a_2\}\right)\le$$

$$min\{a_1,a_2\}+min\{a_2,a_3\}+\cdots+min\{a_{n-1},a_n\}+min\{a_n,a_1\}\le\frac{n}{n-1}min\{a_1,a_2,\cdots,a_n\}$$

Similarly, note that:

$$\min\{a_1,a_2,\cdots,a_k\}\le\frac{1}{2}\min\{a_1,a_2,\cdots,a_{k-1}\}+\frac{1}{2}\min\{a_2,a_3,\cdots,a_k\}$$

Hence, following a similar logic as above,

$$\min\{a_1,a_2,\cdots,a_k\}+\min\{a_2,a_3,\cdots,a_{k+1}\}+\cdots+\min\{a_n,a_1,\cdots,a_{k-1}\}\le$$
$$\min\{a_1,a_2,\cdots,a_{k-1}\}+\min\{a_2,a_3,\cdots,a_k\}+\cdots+\min\{a_n,a_1,\cdots,a_{k-2}\}$$

Which finishes by the induction hypothesis for groups of $k-1$ numbers.

2.3 Application of the Inequalities

In this section, some applications of the above-mentioned inequalities are presented. The inequalities were applied as Robust Inventory Problems. This area focuses on the study of inventory management by a company in order to maximize profits or other objectives.

This paper used the model described in (Ozbay, 2006), chapter 1.2 to model the supply chain and the demands for a specific company. In the model by Ozbay, 2006, the inventory management problem is considered over a discrete

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time period of $t = 1,2,\cdots,T$. x_i denotes the amount of inventory at the start of time i, which can be negative to indicate a shortage. Per unit inventory holding cost is denoted by h_i, which is non-negative by definition. Backlogging cost is represented by b_i, and is non-positive since shortage is denoted by a value less than 0. A production cost c_i is also defined in the paper (by Ozbay). In all of the above notations, a subscript i indicates the value at time i.

The original paper then described three processes that happen at time i:

1. An order of quantity $u_i \geq 0$ is made, increasing the inventory to $x_i + u_i$ and incurring a cost of $c_i u_i$
2. A demand $d_i \geq 0$ is served, decreasing the inventory to $x_{i+1} = x_i + u_i - d_i$
3. At the end of period i, a cost of $\max\{h_i x_{i+1}, b_i x_{i+1}\}$ is paid

To better illustrate why the final cost paid is given by the maximum of the two numbers, the following cases can be considered:

1. The company has a positive inventory (i.e., $x_i \geq 0$). In this case, from $h_i \geq 0$, $b_i \leq 0$, it follows that $h_i x_i \geq 0 \geq b_i x_i$. Hence, $\max\{h_i x_i, b_i x_i\} = h_i x_i$, which is desired since a positive inventory would mean paying an inventory holding cost.
2. Conversely, if the company has a negative inventory (i.e., $x_i < 0$), then $h_i x_i \leq 0 \leq b_i x_i$ (again due to the signs of h_i and b_i). Hence, $\max\{h_i x_i, b_i x_i\} = b_i x_i$, which is again desired since a negative inventory would necessarily pay a backlogging cost.

Hence, the total cost at the end of period T can be written as:

$$\sum_{i=1}^{T}(c_i u_i + \max\{h_i x_i, b_i x_i\}) = \sum_{i=1}^{T} c_i u_i + \sum_{i=1}^{T} \max\{h_i x_i, b_i x_i\}$$

Notice that the second term in the above equation is in the form of sum of maximum of numbers, which allowed previous theorem to be applied to obtain a bound for the second term and the total cost.

To convert the above into the form of previous inequalities, first substitute $a_{2i-1} = h_i x_i$ and $a_{2i} = b_i x_i$. Notice that $a_{2i-1}a_{2i} = h_i x_i b_i x_i = h_i b_i x_i^2 \leq 0$ as $h_i \geq 0$, $b_i \leq 0$.

Let $S = \sum_{i=1}^{T}(h_i x_i + b_i x_i) = \sum_{i=1}^{2T} a_i$. Further define $a_{2T+1} = -S$. Then, $\sum_{i=1}^{2T+1} a_i = 0$.

There are now two cases depending on the sign of S.

Case 1: $S \geq 0$

Hence, by corollary 2.3 on $2T+1$ numbers,

$\frac{2T+1}{2T}\max\{a_1, a_2, \ldots, a_{2T}, a_{2T+1}\} \leq \max\{a_1, a_2\} + \max\{a_2, a_3\} + \cdots + \max\{a_{2T}, a_{2T+1}\} + \max\{a_{2T+1}, a_1\}$ (

Notice that $\sum_{t=1}^{T}(\max\{h_t x_{t+1}, b_t x_{t+1}\}) = \max\{a_1, a_2\} + \max\{a_2, a_3\} + \ldots + \max\{a_{2n-1}, a_{2n}\}$.

Now, the following theorem holds:

Theorem 2.5: *Let $a_1, a_2, \cdots, a_{2n-1}, a_{2n}$ be real numbers such that $a_{2i-1}a_{2i} \leq 0\ \forall\ 1 \leq i \leq n$. Then for $n \geq 2$*

$$\max\{a_1, a_2\} + \max\{a_3, a_4\} + \cdots + \max\{a_{2n-1}, a_{2n}\} \geq$$
$$\max\{a_2, a_3\} + \max\{a_4, a_5\} + \cdots + \max\{a_{2n-2}, a_{2n-1}\} + \max\{a_{2n}, a_1\}$$

(Notice that the above inequality is not symmetric due to the condition being only $a_{2i-1}a_{2i} \leq 0$, not $a_{2i}a_{2i+1} \leq 0$).

Proof. The technique of smoothing can be used

WLOG, $a_1 \geq a_2$.

When $a_{2i-1} \geq a_{2i}$ for all $1 \leq i \leq n$, $LHS = a_1 + a_3 + a_5 + \ldots + a_{2n-1} = RHS$, so the inequality obviously holds.

Else, exist $a_{2i-1} < a_{2i}$. WLOG, let that be a_3 and a_4.

Now split into two cases:

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Case 1.1.

If $a_5 \geq a_6$, swap the order of a_3 and a_4 (i.e., let a_3'=a_4, a_4'=a_5) Notice that the value of LHS do not change. Hence, all is left to show that the value of RHS does not decrease. Note that $a_1 \geq a_2$, $a_1 a_2 \leq 0$, hence $a_2 \leq 0$. Similarly, from $a_3 < a_4$, $a_3 a_4 \leq 0$, it can be obtained that $a_3 \leq 0$. Hence, $\max\{a_2, a_3\} \leq 0$. Let Δ represent the new RHS value subtracting the old RHS value. Note that $a_3 \leq a_4$, $a_5 \geq a_6$. Hence, $a_4 \geq 0$, $a_5 \geq 0$.

$$\Delta = max\ \{a_2, a_3'\} + max\ \{a_4', a_5\} - max\ \{a_2, a_3\} - max\ \{a_4, a_5\} =$$

$$a_3' + a_5 - max\{a_2, a_3\} - max\{a_4, a_5\} = a_4 + a_5 - \max\{a_2, a_3\} - max\ \{a_4, a_5\} \geq$$

$$= a_4 + a_5 - 0 - max\ \{a_4, a_5\} \geq \min\ \{a_4, a_5\} \geq 0$$

Where Δ being positive shows an increase in RHS value after the swap of a_3 and a_4.

Case 1.2. Else $a_5 \leq a_6$

If $a_7 \geq a_8$, then swap a_5 and a_6. The proof concludes using an argument similar to the one above.

Otherwise, $a_7 < a_8$.

But note that such process ($a_{2i-1} \leq a_{2i}$) cannot continue forever, and there must exist *i* such that $a_{2i-1} \geq a_{2i}$. This is due to $a_1 \geq a_2$, which guarantee such *i* exists as the original equation is cyclic. The proof also concludes by using similar reasoning. Hence, by theorem 3.1 and noting that $a_1 = a_{2T} = 0$, $a_{2T+1} = -S \leq 0$,

$$\sum_{t=1}^{T} (\max\{h_t x_{t+1}, b_t x_{t+1}\}) = \max\{a_1, a_2\} + \max\{a_3, a_4\} + \ldots + \max\{a_{2T-1}, a_{2T}\} \geq$$

$$\frac{1}{2}(\max\{a_1, a_2\} + +\max\{a_2, a_3\} + \ldots + \max\{a_{2T-1}, a_{2T}\} + \max\{a_{2T}, a_1\}) =$$

$$\frac{1}{2}(\max\{a_1, a_2\} + +\max\{a_2, a_3\} + \ldots + \max\{a_{2T-1}, a_{2T}\} + \max\{a_{2T}, a_{2T+1}\} + \max\{a_{2T+1}, a_1\} =$$

$$\geq \frac{2n+1}{2n} \max\ \{a_1, a_2, \ldots, a_{2n}, a_{2n+1}\}$$

which provides a lower bound on the minimum possible cost experienced during the *T* days.

Case 2: $S < 0$

Using the result of case 1 and substituting $a_{2i-1} = -h_i x_i$, $a_{2i} = -(-b_i x_i) = b_i x_i$,

$$-S = \max\{a_1, a_2\} + \max\{a_3, a_4\} + \ldots + \max\{a_{2T-1}, a_{2T}\} \geq \frac{2n+1}{2n} \max\{a_1, a_2, \ldots, a_{2n}, a_{2n+1}\}$$

Hence, multiplying both sides by −1 and noting that $\min\ \{a_1, a_2, \ldots, a_n\} = -\ max\ \{-a_1, -a_2, \ldots, -a_n\}$,

$$-S \leq \frac{2T+1}{2T} \min\ \{h_1 x_1, b_1 x_1, h_2 x_2, b_2 x_2, \ldots, h_T x_T, b_T x_T, a_{2T+1}\}$$

As $a_{2i-1} = -h_i x_i, a_{2i} = -b_i x_i$

This provides an upper bound for the possible cost of inventory management.

2.4 Relation to shutdown point

Now, imagine a company A which sells a product B for a marginal profit of p_i on the *i*th day for *i* from 1 to *T*. Each day the company receives a demand $d_i \geq 0$. It has a per unit inventory holding cost of h_i and a backlogging cost $b_i \leq 0$. If the company is currently earning an economic loss, it should shut down if the price is greater than the average variable cost according to most economic textbooks (Samuelson et al., 2021). Although some authors challenge this believe (Sprout 2016; Wang et al., 2004), its usage does not significantly alter the results. To find the average variable cost, one component of that would be the inventory management costs. The definition of a_i below is the same as above. The company should shut down if the price is below the average variable cost, which means that if the marginal

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profit of the ith goods is less than the minimum possible value of inventory management cost (which is given by equation 23), then the company should definitely shutdown. However, if the marginal profit of the ith goods is still larger than or equal to the average variable cost, the company can continue to produce. The above shutdown point i is given by this equations:

$$p_i \geq \frac{2i+1}{2i} \max \{a_1, a_2, \ldots, a_{2i}, a_{2i+1}\}$$

$$p_{i+1} \leq \frac{2i+3}{2i+2} \max \{a_1, a_2, \ldots, a_{2i}, a_{2i+1}\}$$

Conversely, if the marginal profit of the ith goods is greater than or equal to the maximum possible cost of inventory management, then the company should continue to produce. The following table summarized the results from this section.

Shutdown	$p_i < \frac{2i+1}{2i} \max \{h_1x_1, b_1x_1, h_2x_2, b_2x_2, \ldots, h_ix_i, b_ix_i, a_{2i+1}\}$
Continue Producing	$p_i \geq \frac{2i+1}{2i} \min \{h_1x_1, b_1x_1, h_2x_2, b_2x_2, \ldots, h_ix_i, b_ix_i, a_{2i+1}\}$

However, the above result may only be partially applicable depending on the sign of S.

3. Results

This paper presented a previously known inequality and then generalized it to other cases using a similar approach. These results were then applied in a retail inventory management problem with a model of retail inventory management from Ozbay, 2006. Smoothing was then used to prove another theorem, which was necessary for the final result.

The result of the lower bound shows that the total cost needed should be no smaller than a fraction larger than the maximum of all values. This is expected since the expression of the total cost is clearly greater than the maximum of all numbers.

The upper bound shows that the total cost cannot exceed a certain value. An explanation for this would be to note that $S \leq 0$, meaning that left-hand side is small.

4. Limitations

There are a few limitations that could be improved in future works. First, this paper assumes a simple model outlined in section 2.3. In real life scenarios, the problem is often much more complicated and involves more than one supplier or consumer. Hence, this limits the application of the results. Second, the inequalities are unfortunately quite weak. Although they provide exact bounds and were rigorously proven, they may not be very useful in real life.

To solve those limitations, the results from this paper could be integrated into other methods or algorithms. For example, the alpha-beta pruning or the branch-and-cut algorithms all rely on bounding and error estimation for efficiency. More specifically, if the optimal answer will never be found in one case, the algorithm can save time by ignoring that case. Hence, with suitable bounding, such cases can be identified more efficiently, allowing for a better run time.

5. Comparative Analysis

The retail inventory management problem falls under the area of optimization. In optimization, many methods require the use of calculus or analysis, such as the Karush–Kuhn–Tucker conditions (Ghost et al., 2019) or Lagrange multipliers (Sabach et al., 2022). Others uses tools such as linear programming to solve the optimization problem (Vanderbei, 2020). However, polynomials above degree four are not solvable (Żołądek, H. 2000). Hence, purely analytic solutions are rarely used (icurays1, 2017).

In this case, retail inventory management problem often uses algorithms such as machine learning (Gurnani et al., 2021; Gijsbrechts et al., 2022) or other algorithms (Yu et al., 2019). As far as the authors know, there is no purely analytic approach due to the complexity of such problems. Hence, it is difficult to have a meaningful comparison between the algorithms.

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