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# Studies of collective behavior in bounded and exterior domains with repulsion

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**Abstract**

In this note, we perform numerical studies of the Cucker-Smale system for a group of agents confined to a bounded domain or to its exterior. The purpose of this study is to test several hypotheses related to the long-time behavior of the system. Our most definitive observation suggests that in a convex bounded domain, the agents either settle away from the influence of the wall or congregate to a point circulating periodically near the boundary. In the exterior case, we observe a critical ratio between the radius of the flock and the radius of the obstacle that serves as a threshold between two distinctly different limiting behaviors - bouncing back from the obstacle or passing around it.

*Keywords: Alignment, Flocking Behavior, Cucker-Smale system.*

## 1. Introduction

Collective behavior of agents can be used to model many swarming phenomena in biology, such as flocking of birds or milling formations of schools of fish. One collective phenomenon that is abundant in nature is called alignment – convergence of all velocities in the flock to a common vector $v_i \to v_\infty$, as $t \to \infty$. The actual interpretation of the term "alignment" may change depending on the circumstances. For instance in biological systems this may literally mean convergence of a flock to a single direction of motion. In the context of opinion dynamics it may mean reaching a consensus. We refer to Tadmor (2021) and Shvydkoy (2021) for recent overviews and surveys on the subject.

Mathematical models can be used to predict the alignment as well as the rate at which it occurs using the techniques of ordinary or partial differential equations. A particular swarming model that has recently received considerable attention in the mathematical literature is called the Cucker-Smale system as introduced by F. Cucker and S. Smale (2007). The system involves a positive communication function $\phi(r) > 0$ which depends only on the Euclidean distance $r$, monotonely decreasing at infinity, which regulates the strength of connection between each pair of agents $x_i \in R^n$, and is stated as follows

$$x_i' = v_i; \quad v_i' = \frac{1}{N}\sum_{j=1}^{N} \phi\left(\left|x_i - x_j\right|\right)\left(v_j - v_i\right) \qquad (1)$$

Here, $v_i$ is the velocity of the $i$th agent. The main result states that if the communication function does not decay too fast at long range, i.e., there is sufficient connection remaining between even remotely distant agents, the alignment will occur exponentially fast. Here is the precise statement.

**Theorem 1.1** (Cucker and Smale, 2007). *If* $\phi(r) = \frac{H}{1+r^\beta}$, *where* $\beta \le 1$, *then for any initial condition the solution to the system (1) satisfies*

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$$\max_i \left|v_i - v_\infty\right| \le Ce^{-\delta t}$$

*where* $C, \delta > 0$ *depend only on the kernel and the initial condition.*

Much analysis of this original system or systems with various interaction forces was focused on the case when the flock environment is an open space or periodic torus $T^n$. In other words, an environment without boundaries. The main objective of this note is to address the situation when the environment does have boundaries. In particular, we focus on two complementary scenarios: the flock is confined into a bounded region (interior problem) or is located in an open space with obstacles (exterior problem). The motivation for such study comes from an attempt to place the system (1) in a more realistic setting where various obstacles or boundaries are unavoidable. In this situation the swarm is confronted not only with the alignment forces but also the forces that keep agents from colliding with the walls. The basic question we ask is does the flock still reach a collective outcome in presence of boundaries? And what that collective outcome might look like?

Previous research on this problem is scarce. The work of Shu and Tadmor (2020) addresses a somewhat similar situation when the flock is being confined to a bounded region with strong attractive forces, but with no boundaries involved. Here the collective outcome is described by a limiting condition in which the flock aggregates around a pair of phase variables (x,v) satisfying the harmonic oscillator equation

$$x' = v; \quad v' = -x.$$

The work of Bae et. al. (2019) studies the case of a cylindrical domain with specular boundary conditions (perfect reflection). In this situation the flock aggregates near the boundary and with an additional push force along the cylinder does align parallel to the cylinder. The agents of the flock are allowed to collide with the walls in this model.

We will study the situation when the agents are perceptive enough to avoid collisions. To achieve this effect wall interactions will be modeled through a strong repulsive force. We will be interested in testing computationally several conjectures related to the possible collective behavior in these physically reasonable settings. Our findings show that generally in the case of a bounded domain the limiting behavior of a flock depends on whether the domain is convex or not. In the convex case the flock either stalls in the interior away from the boundaries or aggregates to a common position and velocity and behave like a single particle circling around the boundary, somewhat similar to the open space confinement situation of Shu and Tadmor (2020). In a non-convex domain, the system exhibits a much more complicated behavior and may settle on chaotic billiard-like dynamics. Lastly, in the case of the exterior problem, we find that the flock can either bounce back from the boundary, stall near the boundary, or circumvent the boundary and pass the obstacle. These scenarios depend on the ratio between the initial radius of the flock and radius of the obstacle (ball in our case) under a fixed initial velocity. We will find that ratio to be approximately $7.5$.

These and other observations will be detailed in the forthcoming sections. We finish the introduction by introducing precisely the model we are studying.

Let us describe the repulsive force first. It will have a common structure regardless of the situation we consider. The repulsion force starts at zero within one unit length from the boundary, then intensifies and approaches infinity as the agent position closes into the boundary. Otherwise, the force remains smooth and differentiable away from the boundary. This force is the gradient of a potential defined below, where $r$ is the distance from the boundary,

$$U(r) = \frac{(r-1)^2}{r}, if\ 0 < r < 1$$
$$U(r) = 0, \qquad if\ r \ge 1.$$

So, for each agent $x$ the actual force acting on it is defined as follows

$$F(x) = -\nabla_x V, \quad V(x) = U(dist\{x, \partial\Omega\}).$$

The agents will also have an alignment force as in the classical Cucker-Smale system, that gets weaker as the agents are farther away from each other:

$$\phi = \frac{1}{1+r}.$$

Note that this is the weakest kernel in the range of applicability of Theorem 1.1. Thus, the system we study reads as follows:

$$x_i' = v_i; \quad v_i' = \frac{1}{N}\sum_{j=1}^{N} \phi\left(\left|x_i - x_j\right|\right)\left(v_j - v_i\right) + F\left(x_i\right)$$

## 2. Bounded Domains

In this section, we are testing the conjecture that with any initial conditions, the agents inside a circular bounded domain will achieve limiting behavior in one of two ways.

*Conjecture* 2.1. As time goes to infinity one of the following two behaviors occur:

(1) Velocities will align $v_i \to v_\infty$ and particles will aggregate $x_i \to x_\infty$ to a couple $\left(x_\infty, v_\infty\right)$ satisfying the following Hamiltonian system:

$$x'_\infty = v_\infty; \quad v'_\infty = -\nabla V\left(x_\infty\right)$$

(2) The particles come to rest at some point inside the domain and out of range of the repulsive forces of the boundary.

At first, we explored the behavior of three particles starting with random initial conditions in the circular bounded domain. After completing 30 trials we determined that the system would eventually converge to the same end behavior as described in parts (1) and (2) of the conjecture. A sample experiment showing the first possibility is depicted in Figure 1. After a transient period of oscillatory motion the particles aggregated and settled on a looping pattern near the boundary marked by the green curve. Given the circular symmetry of the domain the particles in their limiting state satisfy the harmonic oscillator

$$x'_\infty = v_\infty; \quad v'_\infty = -c\, x_\infty,$$

where the constant "c" is determined by the distance of the limiting circle from the boundary, which is always positive.

The simplest scenario where the agents didn't end up looping around the edge of the boundary was achieved when they started off with symmetric initial conditions. This scenario would still lead to aggregation, but the particles, after a transient oscillatory period, eventually ran out of energy and came to a complete stop in the interior of the disc away from the influence of the wall, as shown in Figure 2.

On a relevant note, we mention that a conjecture similar to our 2.1 was in fact confirmed analytically by Shu and Tadmor (2020) in the case of open space with quadratic confinement potential $U$. The result relies in an essential way on convexity of the potential, which is suggestive of the fact that our conjecture in the convex domain, and hence under convex potential force, may have a similar analytic validation.

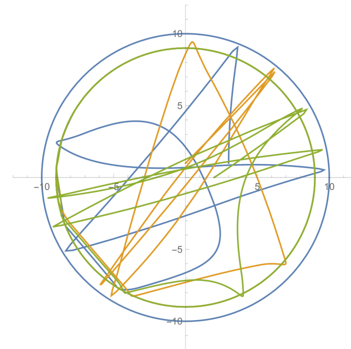

Figure 1: The blue circe is the boundary of the circle, with the boundary repulsive force beginning to take effect one unit inside it. The green circle just inside the boundary of the domain represents the end behavior of the particles. In this simulation, the particles bounced around in random fashion before merging together and settling in a consistent path, which is shown by the green circle just inside the blue boundary circle. The starting conditions were as follows: $x_1 = (3, 1)$, $v_1 = (0, 3)$, $x_2 = (3, 1)$, $v_2 = (3, 3)$, $x_3 = (3, 1)$, $v_3 = (3, 2)$

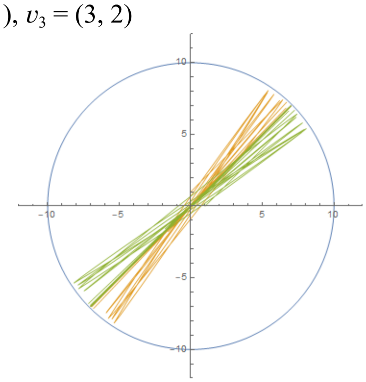

Figure 2: In this case, the blue boundary represents the boundary of the domain, while the end behavior of the particles is represented by the darker green line that makes a 45° angle with both axes. The starting conditions were as follows: $x_1 = (1, 1)$, $v_1 = (3, 3)$, $x_2 = (0, 1)$, $v_2 = (0, 1)$, $x_3 = (1, 0)$, $v_3 = (3, 3)$

We attribute the difference in collective outcomes to the presence of residual total energy. Let us recall that the energy of the system is given by

$$E = \frac{1}{N}\sum_{i=1}^{N} V(x_i) + \frac{1}{2N}\sum_{i=1}^{N} |v_i|^2$$

The energy law can be computed directly from system (1), see (Shvydkoy, 2021):

$$\text{(EL)} \qquad E' = -\frac{1}{N}\sum_{i,j=1}^{N} |v_i - v_j|^2$$

As we can see, the longer the agents remain misaligned, the more energy will be burned by the alignment forces. On the other hand, if the system settles into an aligned configuration before the energy is fully depleted, that residual conserved energy drives the system into a perpetual motion.

We also tested the same system of three particles in an irregular, convex bounded domain as shown in Figure 3. In this irregular domain, the three particles would still aggregate. However, the limiting path is no longer showing any regular pattern. The aggregated system follows a chaotic and unpredictable tranjectory, which exhibits transient periods of traversing the boundary of the domain, especially near the convex parts of it, and other times bouncing off of the boundaries in a billiard-like fashion. The system loses energy if velocities remain misaligned $v_i \neq v_j$ according to (EL). If the misalignment persists for a long time the system comes to a complete stop. If the system aggregates and aligns with residual energy left, the agents settle into a perpetual billiard-like motion.

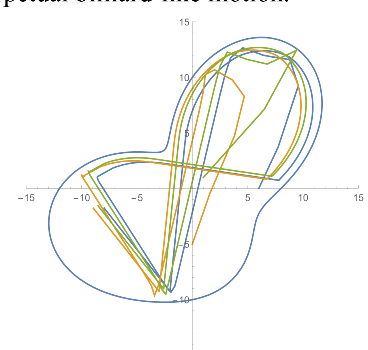


Figure 4: The blue line represents the boundary of the domain. Starting conditions:$x_1$ = (6, 0), $v_1$ = (4, -2), $x_2$ = (0, -5), $v_2$ = (0, 3), $x_3$ = (1, 1), $v_3$ = (5, 5). The particles align and settle into a perpetual billiard-like dynamics. Domain is given by r = 10 + 4sin(2θ) - 2cos(3θ)

## 3. Conclusion of the Bounded Domains case

The results of our computations indicated rather convincingly that the limiting behavior of the Cucker-Smale system in a bounded convex domain follows one of the two outcomes described under Conjecture 2.1. The system settles either into an oscillatory motion near the boundary, or to an equilibrium in the interior of the domain. In the former case the energy of the system remains positive and the agents move perpetually. In the latter case the energy burns down to zero and the agents come to a complete stop.

In the case of a non-convex domain we found that the limiting behavior does not fall under the general conclusions of Conjecture 2.1. The system undergoes a period of chaotic motion before settling into an equilibrium state inside the domain if misalignment persists for a long time. Otherwise, the system settles into a perpetual chaotic motion. We believe that in non-convex domains the complete description of collective outcomes depends on the particular shape of the domain and is no longer universal as in the convex case. We anticipate that any attempt to classify the behavior in this setting would rely on the theory of dynamical systems, which we plan to investigate in future studies.

## 4. Exterior Domains

With exterior domains, we would have a group of particles with initial velocities $v_0$ = (1, 0) and starting positions at $x = -40$, and a number of exterior domains in the path of the particles to test out how the particles would react. We conjectured that there would be one of the three outcomes from these experiments.

*Conjecture* 3.1. The simulation, as time goes to infinity, will follow one of the following three behaviors:

(1) The particles successfully pass the exterior domains and continue with a positive velocity in the x direction.

(2) The particles rebound from the obstacles and aggregate to a negative velocity in the x direction.

(3) The particles are almost completely stopped by the obstacles, and continue with a near-zero velocity in the x-direction.

### 3.1 *One Obstacle*

First, we tested this conjecture on a case with the exterior of a ball. We had to flip the boundary force equation so that it acted outside the boundary instead of inside. We increased the number of particles to ten, and spread them out evenly on the range $[-10, 10]$. We made the radius of the ball a variable that could be easily changed to test which different radii result in the particles rebounding backwards or going around. Figures 5,6, and 7 show examples of various scenarios.

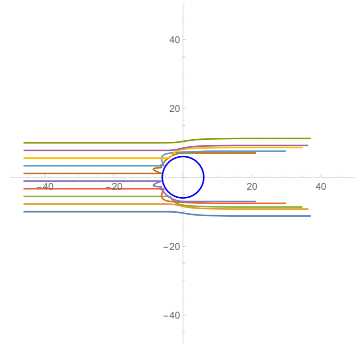


Figure 5: Particles successfully managed to pass around the obstacle, with the radius of the tree being 6.

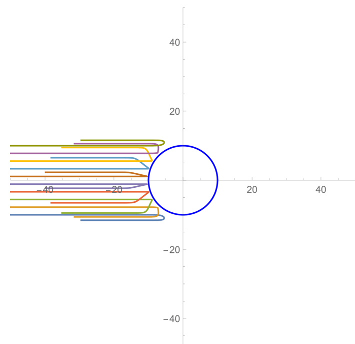


Figure 6: Particles did not pass around the obstacle and instead got turned around, with the radius of the obstacle being 10.

After testing the simulation 40 times and making the corresponding adjustments, we figured out the exact radius of the obstacle such that the particles wouldn't pass by or get rebounded, instead come to an almost complete stop. This threshold value of the radius just about 7.5, with the particles ending with slight forward velocities with magnitudes around $10^{-6}$ when the radius is exactly 7.5. Figure 7 shows how the particles stalled and slowed down drastically.

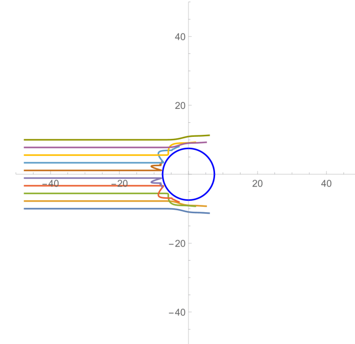


Figure 7: Particles stalled and came to an almost complete stop, with the radius of the obstacle being 7.5.

### 3.2 *Multiple Obstacles*

Another model we tested out using exterior boundaries was a configuration of many smaller obstacles in a random distribution, reminiscent of a forest. The forest would consist of 30 trees (boundary circles with radius one) on random integer coordinates in a 45 by 60 rectangle. To make sure that the trees wouldn't overlap, we would produce a random integer on the range -10 to 10, then would be multiplied by 3 in order to produce the desired random $y$ coordinate, with a similar procedure used to produce the $x$ coordinate. The three particles we used would be placed far left, at $x = -40$, and have random $y$ values. All the particles would have initial velocity $v = (1, 0)$. Because we had 30 random boundaries, each with their own repulsive force, we had to have a sum of all the forces in the differential equations of each particle,

$$F = -\sum_{i=1}^{30} \nabla V_i.$$

We ran the simulation many times, and the particles would sometimes manage to get through the trees, and other times not, as seen in Figures 8 and 9.

We found that whether the particles pass through the "forest" or not was highly dependent on the random starting positions. With the density of obstacles and starting values as described in the beginning, the particles did not pass through the forest most of the time. However, there were still multiple cases where they managed to successfully do so as seen in Figure 9.

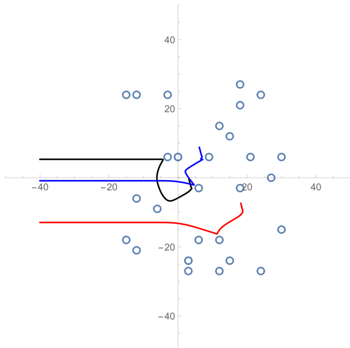


Figure 8: Particles did not manage to get through the trees.

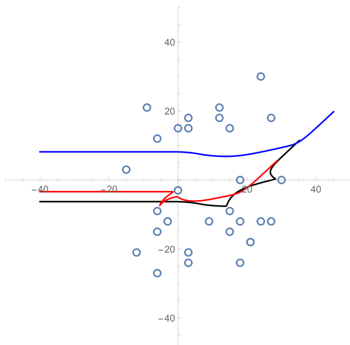


Figure 9: Particles successfully pass through the trees.

## 5. Conclusion of the Exterior Domains case

Through the testing of the behavior of the agents in presence of one exterior domain and multiple exterior domains, we can conclude that our hypothesis as stated in Conjecture 3.1 is correct. In both scenarios we tested (one obstacle vs. many), the particles always followed one of the three behaviors as outlines in Conjecture 3.1: the particles get around/through and end with a positive x-velocity, the particles get stopped and continue with a near-zero velocity, or the particles are rebounded and continue with a negative x-velocity.

For the case of one circular exterior domain, after testing the simulation 40 times and adjusting the radius of the obstacle, we found that the critical ratio between the range on which the particles are distributed and the diameter of the obstacle to be 7.5. This threshold value was found to be robust. It is independent of the actual characteristic dimensions of the system as long as the ratio remains unchanged. However, it still varies with the initial energy of the system. This observation calls for a more rigorous analysis of universality of the passing threshold value, which we plan to undertake in near future.

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