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A Brief Introduction to Wormhole Research

Luanna Veroneze Quinalha[1 *]

[1]Colégio Marista Anjo da Guarda, Curitiba, Paraná, Brazil

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Abstract

Wormholes have been an enduring fascination of the science fiction community despite being marginalized by the physics community since their inception. Wormhole research, however, has seen an increase in attention from theoretical physicists following recent results which suggest that traversable wormhole geometries sourced by massless charged fermion fields could be embedded in the Standard Model at length scales below the electroweak scale (Maldacena, et al., 2018). This article reviews the simplest wormhole solution, the Ellis-Bronnikov-Morris-Thorne wormhole, in a manner accessible to students who have studied general relativity at the advanced undergraduate level. The energy requirements and physical plausibility for such a solution are discussed. Some recent progress in wormhole research, numerical and analytic, is briefly reviewed. Despite intriguing theoretical advancements, the results presented here do not suggest that traversable wormholes will become technologically viable in the foreseeable future.

Keywords: Wormholes, Ellis Wormhole, Traversable Wormholes, General Relativity, Faster-than-Light Travel

1. Introduction

Einstein's theory of general relativity (GR) relates the density and flux of energy and momentum in spacetime to the curvature of spacetime through Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, G is Newton's gravitational constant, and $T_{\mu\nu}$ is the stress- energy tensor. Equation 1 expresses the mathematical relation through which matter and radiation fields source the dynamical evolution of the spacetime curvature, and curvature in turn sources the dynamical evolution of the matter and radiation fields.

Shortly after Einstein's publication of his theory of general relativity, Schwarzschild (1916) presented a spherically symmetric solution to the vacuum field equations. It then became common to think of spacetime manifolds with “point singularities" as a model for massive particles within GR. However, such models were not geodesically complete: they failed to provide a complete history for the particles and light rays which met the singularity. Einstein and Rosen (1935) tried to do away with the singularity by connecting two Schwarzschild exteriors through the Einstein-Rosen bridge. Their idea was to explain the atomic nature of matter by imagining elementary particles as topological holes in space. However, their geometry had a degenerate metric and was also geodesically incomplete (Hawking and Ellis, 1973) (note: in 1916, Ludwig Flamm arrived at a construction with two asymptotically flat ends, but he did not contemplate the possibility of the Einstein-Rosen bridge).

Some years later, Kruskal (1960) and Fronsdal

* Corresponding Author
luanna.v.quinalha@gmail.com

Advisor: Seth Olsen
srolsen@princeton.edu

(1959) independently showed that the maximal analytic extension (one in which every geodesic originating from an arbitrary point has infinite length in both directions or ends at a singularity that cannot be removed by a coordinate transformation) of Schwarzschild's metric has a hole in it, associated with the Schwarzschild horizon. Wheeler (1955) called this hole a "wormhole". He and Fuller also discovered that this topology does not violate causality, since the throat pinches off before a light signal can be transmitted between the two ends. Even though the Schwarzschild wormhole is singularity-free, it is geodesically incomplete due to its horizon.

Ellis and Bronnikov were the first to independently find a geodesically complete wormhole manifold in 1973. In 1988, Morris and Thorne (1988a), unaware of their discoveries, published a duplicate of the same solution as a tool for teaching general relativity. In a different paper, they also explored its implications for time travel (Morris and Thorne, 1988b).

At that time, there was no precise mathematical definition to distinguish wormholes from other similar geometries. For a long time after Morris and Thorne's publication, wormholes were viewed as topological objects connecting two asymptotically flat regions through a throat. Currently, however, a refined understanding of wormholes does not rely on topology (a wormhole solution can be topologically trivial), with the wormhole throat being locally defined, for the static case, in terms of a hypersurface of minimal area subject to a "flare-out" condition (Hochberg and Visser, 1997). In that manner, the mathematical flaring-out condition from Morris and Thorne is generalized. Defining dynamic wormholes is more complex: to accommodate that case, a wormhole throat is defined as a marginally anti-trapped surface (a closed two-dimensional spatial hypersurface such that one of the two future-directed null geodesic congruences orthogonal to it is just beginning to diverge). The divergence property of the null geodesics at the marginally anti-trapped surface generalizes the "flare-out" condition (Hochberg and Visser, 1998).

This article aims to review the Ellis-Bronnikov-Morris-Thorne (EBMT) wormhole and to present recent numerical and analytic progress in the field in a manner accessible to beginning graduate and advanced undergraduate students in physics and astronomy. Therefore, it bridges the gap between an advanced field of research and advanced undergraduate level students and summarizes years of research in a condensed volume. Since reviews on advanced topics such as wormholes tend to be very long, a primary goal here is to be as concise as possible, resulting in only a small amount of attention being given to any one topic. Rather than describing all the relevant mathematical and physical background in this work, pedagogical sources are cited for in-depth follow-up. The topic receiving the most focus is the physical plausibility of traversable wormholes, and recent wormhole research is reviewed much more briefly. Although the results presented here do not suggest that traversable wormholes will become technologically plausible in the foreseeable future, wormhole research has seen significant theoretical advances in recent years.

The article is structured as follows. Section 2 is the focus of the review, analyzing the simplest traversable wormhole solution in detail and studying the EBMT wormhole as an example. Section 2.1 begins by presenting a set of traversability requirements to be imposed on the degrees of freedom in the general spherically symmetric, time-independent spacetime metric. Section 2.2 presents some mathematical derivations useful for interpreting the geometric consequences of traversability requirements. Section 2.3 discusses the consequences of such requirements on matter-fields in terms of the stress-energy tensor of a perfect fluid. Section 2.4 explores the consequences of traversability in a simple static example: the EBMT wormhole. Wormhole dynamics are not discussed in detail. Section 3 is an overview of recent advances in wormhole research. Section 3.1 discusses the challenges faced when defining wormholes more generally and refers the reader to historical progress in the static case. Section 3.2 describes the generalization to dynamic wormholes and Section 3.3 discusses the connection between wormhole and black hole geometries. Section 3.4 is a brief presentation of traversability requirements not covered in Section 2: stability and assembly. Section

3.5 discusses rotating wormholes and Section 3.6 refers the reader to recent progress on wormhole geometries in beyond-GR theories.

2. Ellis-Bronnikov-Morris-Thorne Wormhole

In this section, the general metric for the simplest traversable wormhole is derived and its topological and energetic properties are considered. The EBMT wormhole, a specific version of this general metric, is then explored using some of the results obtained for the general solution.

2.1 Setup for the simplest traversable wormhole

Since the goal is to find the simplest possible traversable wormhole solution, a static and spherically symmetric metric which satisfies the following traversability requirements (Morris and Thorne, 1988a):

1. The solution must have a throat connecting two asymptotically flat regions of spacetime
2. There should be no horizon
3. The tidal gravitational forces on a traveler should be bearably small
4. Travelers must be able to cross the wormhole in a reasonable amount of proper time as measured both by themselves and by observers outside the wormhole
5. The matter and fields that generate the wormhole's spacetime curvature must have a physically reasonable stress-energy tensor
6. The solution should be perturbatively stable
7. It should be possible to assemble the wormhole

is desired. The general spherically symmetric, time-independent spacetime metric can be expressed as

$$ds^2 = -e^{2\Phi}c^2dt^2 + \frac{dr^2}{1-\frac{b}{r}} + r^2(d\theta^2 + sin^2\theta d\varphi^2) \quad (2)$$

where $\phi = \phi(r)$, $b = b(r)$ are two arbitrary functions of radius to be constrained by the desired properties listed above. More specifically, $b(r)$, the shape function, determines the spatial shape of the wormhole, while $\phi(r)$, the redshift function, encodes the time dilation relative to asymptotic observers as a function of the radius and determines the gravitational redshift.

In the following sections, the consequences of requirements 1, 2, and 3 will be explored. Interested readers may learn more about requirement 4 in Morris and Thorne's original article (Morris and Thorne, 1988a). Requirement 5 is explored in a “backward" manner - instead of being imposed and having its mathematical consequences explored, information is first obtained about the matter and fields needed to generate the wormhole so that their plausibility can then be discussed. Requirements 6 and 7 are outside the scope of this article, although they are briefly discussed in the Recent Progress section.

2.2 Geometric Consequences of Traversability

Here, the first steps towards constructing an embedding diagram for the EBMT wormhole are taken. Those steps will allow the translation of the traversability requirements listed in 2.1 into constraints on the spacetime metric and its geometric characteristics. Specifically, the equatorial slice $\theta = \pi/2$ of the metric will be visualized at a fixed moment of time t. The line element for such a slice is

$$ds^2 = (1 - b/r)^{-1}dr^2 + r^2d\varphi^2 \quad (3)$$

The goal here is to construct a $2D$ surface with the same geometry as this slice, embedded in $3D$ Euclidean space. The line element for such an embedding space is, in cylindrical coordinates, given by

$$ds^2 = dz^2 + dr^2 + r^2d\varphi^2 \quad (4)$$

which, when considering the surface $z(r)$, can be rewritten as

$$ds^2 = [1 + (\frac{dz^2}{dr})]dr^2 + r^2d\varphi^2 \quad (5)$$

For this to correspond to the chosen wormhole slice in Equation 3, (r, φ) for the wormhole is identified with (r, φ) for the embedding space. Once that is imposed, all that is left to do is require

$$\frac{dr}{dz} = \pm (\frac{r}{b(r)} - 1)^{-1/2} \quad (6)$$

Now think of the coordinate r as the distance from the wormhole origin at which a two-sphere has area $A(r) = 4\pi r^2$. Additionally, define l, a signed radial coordinate whose absolute value on each side of the wormhole throat is equal to the proper radial distance from the center of the wormhole, as (Morris and Thorne, 1988a):

$$l(r) = \pm \int_{b_0}^{r} dr/(1 - b(r)/r)^{1/2} \qquad (7)$$

This coordinate covers $(-\infty, \infty)$, connecting the asymptotically flat region (where $A(r) \rightarrow \infty$) on one side of the wormhole with the same geometry on the other side. As l goes from one side of the wormhole to the other, $A(r)$ will decrease to a minimum before expanding again. Therefore, the wormhole's throat is naturally defined by the minimum of $A(r)$, or equivalently by the condition that $dr/dl = 0$ at the throat. Here, the throat is taken to be at $l = 0$.

From this definition, it follows that $dl/dr \rightarrow \infty$ at the throat. This implies $b(r) = r$. Away from the throat, $b(r) < r$. In other words,

$$1 - \frac{b}{r} \geq 0 \; throughout \; spacetime \qquad (8)$$

The wormhole throat may also be defined as the point with minimal radius $r = b_0$ at which its embedded surface (Figure 1) is vertical. Therefore, at the throat, $dz/dr \rightarrow \infty$, which is consistent with the first definition and with Equation 6. Far from the throat, the wormhole must connect two asymptotically flat regions (traversability requirement 1), so $\frac{dz}{dr}$ must approach zero as $l \rightarrow \mp\infty$.

$$\lim_{l \rightarrow \mp \infty} \frac{dz}{dr} = 0 \qquad (9)$$

Equations 6 and 7 imply

$$\frac{dz}{dl} = \pm \sqrt{\frac{b}{r}} \qquad (10)$$

$$\frac{dr}{dl} = \pm \sqrt{1 - \frac{b}{r}}$$

Finally, traversability requirement 2 states that the wormhole should be horizon-free. A horizon is a surface in which $g_{00} = -e^{2\phi} \rightarrow 0$ (Vishveshwara, 1968). Therefore, demanding no horizons means

$$\phi(r) \; is \; finite \; everywhere \qquad (11)$$

In Section 2.4, some of these results are used to create the embedding diagram of the EBMT metric (a simple example of the general solution in Equation 2). In section 3, some of the formal aspects of the throat and wormhole definitions which were simplified for our use here are reviewed, elaborating on the formal definitions presented in the Introduction.

2.3 Stress-Energy Tensor

Now that the geometric characteristics of our solution have been analyzed, it is only natural to ask what kind of material is needed to physically create it. In this section, the impact of the constraints placed on the metric so far on the type of stress-energy required for the simplest traversable wormhole is studied.

<u>*Meaning of Each Term*</u>

According to Birkhoff's theorem, only one kind of spherically symmetric vacuum wormhole is allowed by the Einstein field equations: a Schwarzschild wormhole (see section 5.2 of Carroll, 2004). As mentioned in the Introduction, such a wormhole is non-traversable. Therefore, the simplest traversable wormhole should have a non- zero stress-energy tensor.

The generalized metric in Equation 2 can be used to represent the only non-zero components of the Einstein tensor as:

$$G_{tt} = b'/r^2 \qquad (12)$$

$$G_{rr} = -\, b/r^3 + 2(1 - b/r)\phi'/r$$

$$G_{\theta\theta} = G_{\varphi\varphi} = (1 - \tfrac{b}{r})(\phi'' - \tfrac{br-b}{2r(r-b)}\phi' + (\phi')^2 + \tfrac{\phi'}{r} - \tfrac{br-b}{2r^2(r-b)})$$

Since the stress-energy tensor is proportional to the Einstein tensor, the only non-zero terms of $T_{\mu\nu}$ will be T_{tt}, T_{rr}, $T_{\theta\theta}$, and $T_{\varphi\varphi}$. Because a static observer's basis vectors are being used, each component has an interpretation in terms of the measurements such an observer might make:

$$T_{tt} = \rho(r)c^2,\; T_{rr} = -\,\tau(r),\; T_{\theta\theta} = T_{\varphi\varphi} = p(r) \qquad (13)$$

where $\rho(r)$ is the total mass-energy density, $\tau(r)$ is the tension per unit area in the radial direction, and $p(r)$ is the pressure in lateral directions. This is the stress-energy tensor for a perfect fluid, and the consequences for the nature of this fluid that result from the wormhole geometry and traversability requirements will now be explored.

Results from the Einstein Field Equations

The first step to imposing those consequences is to express the functions that appear in the non-zero terms of the stress-energy tensor (ρ, τ, and p) in terms of the functions used to express the metric in Equation 2 (b and ϕ). The Einstein field equations (Equation 1), when evaluated with the Einstein tensor in Equation 12 and the stress-energy tensor in Equation 13, become

$$b' = 8\pi G c^{-2} r^2 \rho \quad (14)$$
$$\phi' = (-8\pi G c^{-4} \tau r^3 + b)/[2r(r-b)]$$
$$\tau' = (\rho c^2 - \tau)\phi' - 2(p+\tau)/r$$

which can be rewritten as

$$\rho = b'/(8\pi G c^{-2} r^2) \quad (15)$$
$$\tau = [b/r - 2(r-b)\phi']/[8\pi G c^{-4} r^2] \quad (16)$$
$$\rho = (r/2)[(\rho c^2 - \tau)\phi' - \tau'] - \tau \quad (17)$$

This form will allow $b(r)$ and $\phi(r)$ to be tailored to make a nice wormhole. Equation 15 and the chosen $b(r)$ will be used to find $\rho(r)$. Equation 16 and the chosen functions of $b(r)$ and $\phi(r)$ will be used to get $\tau(r)$. Finally, Equation 17 and the chosen $\phi(r)$, together with the results obtained for $\rho(r)$ and $\tau(r)$, will be used to determine $p(r)$.

Constraints at the Throat

Now that the stress-energy components have been properly expressed, one may begin analyzing the energetic requirements of the wormhole. The most severe constraints on the wormhole's shape function, and, therefore, on its matter-fields content, are at the throat.

There, $r = b = b_0$ and $(r - b)\phi' \rightarrow 0$. Together with Equation 16, these expressions imply

$$\tau_0 = \frac{1}{8\pi G c^{-4} b_0^2} \approx 5 \times 10^{41} \frac{dyn}{cm^2}\left(\frac{10m}{b_0}\right)^2 \quad (18)$$

which is an enormous tension. For $b_0 \sim 3km$, the tension would have the magnitude of the pressure at the center of the most massive of neutron starts ($\sim 10^{37} dyn/cm^2$).

In the neighborhood of the throat, this tension can be further investigated by defining the dimensionless function

$$\zeta = \frac{\tau - \rho c^2}{|\rho c^2|} = \frac{b/r - b' - 2(r-b)\phi'}{|b'|} \quad (19)$$

Traversability requirement 1 states that the wormhole should connect two asymptotically flat regions of spacetime. Therefore, at the throat, the embedding surface should flare outward. Mathematically, $\frac{d^2r}{dz^2} > 0$ at or near r = b. To explore this constraint, invert Equation 6:

$$\frac{dr}{dz} = \pm\left(\frac{r}{b(r)} - 1\right)^{1/2} \quad (20)$$

Next, differentiate this to obtain one version of the flaring-out condition:

$$\frac{d^2r}{dz^2}(r \approx b) = \frac{b - b'r}{2b^2} > 0 \quad (21)$$

By rewriting ζ as

$$\zeta = \frac{2b^2}{r|b'|}\left(\frac{d^2r}{dz^2}\right) - 2(r-b)\frac{\phi'}{|b'|} \quad (22)$$

and using the finiteness of ρ and b' and the fact that $(r - b)\phi' \rightarrow 0$ at the throat, the flare-out condition can be rewritten as

$$\zeta_0 = \frac{\tau_0 - \rho_0 c^2}{|\rho_0 c^2|} > 0 \; \textit{at or near the throat, } r = b = b_0 \quad (23)$$

where τ_0 and ρ_0 stand for $\tau(r = b_0)$, the tension per unit area in the radial direction at the throat, and $\rho(r = b_0)$, the total mass-energy density at the throat, respectively.

Exotic Material

The constraint $\tau_0 > \rho_0 c^2$ is troublesome, and material with this property is called exotic. If an observer moves through the throat sufficiently fast (close to the speed of light), they will measure a negative mass-energy density, which is a signature of a pathological theory. It can be useful to classify spacetimes and their corresponding stress-energy tensors according to which energy conditions they obey or violate. Among the most important conditions are (see section 4.6 of Carroll, 2004):

1. The Weak Energy Condition (WEC), which state that $T_{\mu\nu}\, t^{\mu}t^{\nu} \leq 0$ for all timelike vectors t^{μ}.
2. The Null Energy Condition (NEC), which states that $T_{\mu\nu}\, l^{\mu}l^{\nu} \geq 0$ for all null vectors l^{μ}.
3. The Dominant Energy Condition (DEC), which states that $T_{\mu\nu}t^{\mu}t^{\nu} \leq 0$ for all timelike vectors t^{μ} (the WEC is satisfied) and that $T^{\mu\nu}t\mu$ is nonspacelike for all timelike t^{μ}.
4. The Null Dominant Energy Condition (NDEC), which is the DEC for null vectors ($T^{\mu\nu}l_{\mu}$ is nonspacelike for all null vectors l^{μ}).
5. The Strong Energy Condition (SEC), which states that $T_{\mu\nu}t^{\mu}t^{\nu} \geq \frac{1}{2}T^{\lambda}_{\lambda}t^{\sigma}t_{\sigma}$ for all timelike t^{μ}.
6. The Averaged Null Energy Condition (ANEC), which states that $\int_{c} T_{\alpha\beta}l^{\alpha}l^{\beta}d\lambda \geq 0$ for every integral curve C of the null vector field l; this can be thought of as a less strict version of the NEC, where only the average of $T_{\alpha\beta}l^{\alpha}l^{\beta}$ has to be positive.

These are important when discussing the plausibility of wormhole geometries: it is generally expected for the NEC, or at least the ANEC, to be satisfied. All energy conditions will be violated by $\rho > \tau c^2$. It is true that, under certain circumstances, quantum fields can violate all the energy conditions. Even so, based on current observations, it seems unlikely that such exotic material would be viable at the macroscopic scales required to build the simplest traversable wormhole solution studied here.

Therefore, physicists try to minimize the need for such material. For that purpose, there are three main strategies (Morris and Thorne, 1988a):

1. Use exotic material throughout the wormhole but insist its density fall off quickly moving away from the throat
2. Use exotic material as the only source of curvature but cut it off completely at some radius RS
3. Relegate exotic material to a tiny central region around the throat and surround it with normal matter

Method 1 produces solutions in which the exoticity ζ is positive everywhere from the throat to the asymptotically flat regions it connects. As such, it is the least pleasing strategy since it is the one that least confines exotic matter and is thus least successful in limiting its use. Method 2 works better, but it still requires exotic matter in a larger region than Method 3 does. Confining the use of exotic matter to the immediate vicinity of the throat allows solutions in which the exoticity of that region is produced by the Casimir effect (Morris and Thorne, 1988b). Therefore, the last strategy produces more plausible solutions and is preferred. For examples of solutions that utilize each strategy, see the Appendix of Morris and Thorne, 1988a.

Applying each method is outside the scope of this article, but the reader should keep in mind that the need for exotic matter presents a strong obstacle to a traversable wormhole's technological plausibility. Two further obstacles are briefly presented in section 3.4.

Global Constraints

Above, the local constraints which must be obeyed at the wormhole's throat were found and shown to be problematic. One may use the third traversability requirement to find global constraints, which must be obeyed not only at the throat but throughout the wormhole. It follows from the third traversability requirement that the tidal acceleration experienced by the traveler should not be much greater than around 1 Earth gravity, which implies (Morris and Thorne, 1988a)

$$\left|R_{1010}\right| = \left|\left(1 - \frac{b}{r}\right)\left(-\phi'' + \frac{b'r-b}{2r(r-b)}\right)\phi' - (\phi')^2\right| \frac{g}{c^2\times 2m} \simeq \frac{1}{(10^{10}cm)^2} \tag{24}$$

$$\left|R_{2020}\right| = \left|\frac{\gamma^2}{2r^2}\left[\left(\frac{\upsilon}{c}\right)^2\left(b' - \frac{b}{r}\right) + 2(r-b)\phi'\right]\right| \frac{g}{c^2\times 2m} \simeq \frac{1}{(10^{10}cm)^2} \tag{25}$$

The radial tidal constraint (Equation 24) constrains the metric coefficient ϕ, and is most easily satisfied by requiring $\phi' = 0$. The lateral tidal constraint (Equation 25) constrains the traveler's speed υ.

It is worth reinforcing that it is not yet known how the material needed to form a wormhole would couple to the human body. Therefore, even if it is obtained, and the constraints above are satisfied, the risk remains that it would make the traveler's journey very uncomfortable, perhaps even impossible.

2.4 EBMT Wormhole

Now that the general metric in Equation 2 has been analyzed, the focus of the article will switch to a specific case of that metric, first discovered independently by Ellis and Bronnikov:

$$ds^2 = -c^2dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + sin^2\theta d\varphi^2) \quad (26)$$

Properties and Diagram

The study of the metric in Equation 26 will begin with an analysis of its geometrical characteristics and the creation of an embedding diagram. By looking at the metric, one can immediately notice it's time-independent: no part of it involves the time coordinate, so the geometry does not depend on time at all. One can similarly notice spherical symmetry by observing the term $(b_0^2 + l^2)(d\theta^2 + sin^2\theta d\varphi^2)$, proportional to the metric for a 2-sphere. These properties should come as no surprise since this is a special case of a general static and spherically symmetric metric. However, it is useful to verify them as an exercise.

As $l \rightarrow \pm\infty$, $r \rightarrow +\infty$. Therefore, if the metric were written with a dr^2 component, its coefficient would tend to 1, and there would be two asymptotically flat regions (requirement 1 is satisfied). Additionally, this metric has $\phi = 0$, which is finite and implies no horizons (requirement 2 is satisfied).

Following the work done in section 2.2, it is possible to create an embedding diagram for this wormhole. The metric for the embedding space can be expressed in cylindrical coordinates, as in Equation 4. For the EBMT wormhole, the two-surface

$$z(r) = \pm\, b_0 ln(r/b_0 + \sqrt{(r/b_0)^2 - 1} \quad (27)$$

with $l = \pm\, (r^2 - b_0^2)^{\frac{1}{2}}$ will have the same geometry as the 2-surface $\theta = \pi/2$, $t = const.$ in the spacetime of Equation 2 (Morris and Thorne, 1988a). The two-surface in Equation 27 is plotted in Figure 1.

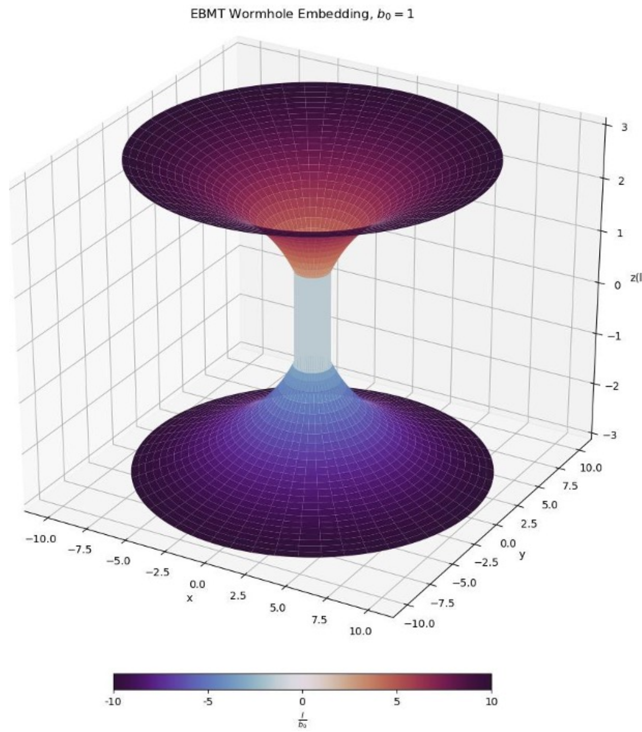

Figure 1. Embedding diagram for an EBMT wormhole with minimum radius b_0 = 1. The wormhole throat (vertical region in which $\frac{dz}{dr} \rightarrow \infty$) connects two asymptotically flat regions of spacetime.

Stress-Energy Required

Now, the question is: what is the stress-energy necessary for this geometry to exist? Using the unit basis vectors e_t, e_r, e_θ, e_φ pointing along the t, r, θ, φ directions, the only nonzero components of the Riemann curvature tensor are $R_{\theta\varphi\theta\varphi} = -R_{l\theta l\theta} = -R_{l\varphi l\varphi} = b_0^2/(b_0^2 + l^2)^2$.

Therefore, the stress-energy tensor is $-T^{tt} = -T^{ll} = T^{\theta\theta} = T^{\varphi\varphi} = (\frac{c^4}{8\pi G})b_0^2/(b_0^2 + l^2)^2$. This is similar to the electromagnetic stress-energy for a static point charge, but with negative energy density (exotic material). In other words, the specific EBMT solution requires, like the general metric, matter-fields which have never been observed or predicted in significant amounts.

Freely Falling Observer

Finally, one can analyze the motion of a potential freely falling observer through the wormhole to find out whether traversability requirement 3, which requires the tidal forces acting on a human traveler to be bearable, is met. An observer falling freely and radially would fall along $\theta = constant$, $\varphi = constant$, by definition. Their equation of motion may be obtained by solving the geodesic equation for $\mu = l$. The result is that

$$\frac{d^2x^l}{dt^2} = 0 \rightarrow \frac{dx^l}{dt} = \nu = constant \qquad (28)$$

Therefore, they fall along $\theta = constant$, $\varphi = constant$, $l = \upsilon t$, where $\upsilon = constant < c$.

Additionally, a Lorentz transformation may be applied to find the basis vectors of the observer's local Lorentz frame: $e_0 = \gamma e_t + \left(\frac{\nu}{c}\right)\gamma e_l$, $e_1 = \gamma e_l + \left(\frac{\nu}{c}\right)\gamma e_t$, $e_2 = e_\theta$, $e_3 = e_\varphi$, where $\gamma = \frac{1}{\sqrt{1-\nu^2/c^2}}$ is the lorentz factor.

Lastly, it is possible to calculate the tidal gravitational force such an observer would experience. In the observer's local Lorentz frame, the only nonzero, tide-producing components of the Riemann tensor, R_{j0k0}, are $R_{2020} = R_{3030} = -\left(\frac{\nu}{c}\right)^2\gamma^2 b_0^2/(b_0^2 + l^2)^2$. Because tidal accelerations are proportional to these components, they vanish as $\upsilon \rightarrow 0$. As such, the third traversability requirement can be met by a traveler with a sufficiently small speed.

This concludes the study of the simplest possible traversable wormhole solution.

3. Recent Progress

Here, the main achievements and contributions to wormhole research since Morris and Thorne's original article are summarized. Though we cannot include every work, discovery, or name relevant to the evolution of the field, this section attempts to provide the information needed for a general overview. The goal is to present recent, significant results and interpret them in terms of physical plausibility while considering what has been covered in sections 1 and 2.

3.1 Defining a (Static) Wormhole

In the Introduction, the definition of a wormhole was briefly mentioned. The context in which it was developed is detailed below.

For a long time, wormholes were viewed as topological objects connecting two asymptotically flat regions through a throat. However, there exist classes of geometries which one might reasonably want to include in the "wormhole" category, and which have no asymptotically flat regions or have a trivial topology. The only difference between the type of wormhole generally imagined (one serving as a "bridge") and these other geometries (topologically trivial ones) arises at the level of global geometry and topology. Therefore, the fundamental property defining a wormhole must be found at the local level. Specifically, a wormhole throat can be locally defined in terms of a hypersurface of minimal area subject to a "flare-out" condition (Hochberg and Visser, 1997). In that manner, asymptotically flat regions are not required, and the mathematical flaring-out condition from Morris and Thorne is generalized.

Having a general wormhole definition allowed Hochberg and Visser (1997) to form generalized theorems regarding violations of the energy conditions. Specifically, they found that the null energy condition, when weighted and integrated over the wormhole throat, must be violated. This result can be seen as complementary to the topological censorship theorem (Friedman, et al., 1993) and it generalizes the energetic obstacles associated with wormholes discussed in section 2.3.

Although Hochberg and Visser initially developed this definition assuming a static wormhole, they soon extended their results to dynamic situations.

3.2 Dynamic Wormholes

To accommodate dynamic wormholes, Hochberg and Visser (1998) had to add some mathematical complexity to their definition. They define a wormhole throat to be a marginally anti-trapped surface (a closed two- dimensional spatial hypersurface such that one of the two future-directed null geodesic congruences orthogonal to it is just beginning to diverge). The divergence property of the null geodesics at the marginally anti-trapped surface generalizes the "flare-out" condition.

As was true in the static case, this generalized definition led to generalized conclusions regarding energy conditions. It was shown that dynamic wormholes have two throats, one for each direction of travel along them. For each of them, the NEC is either violated or on the verge of being violated (Hochberg and Visser, 1998).

At this point, it is clear that wormholes inevitably violate the NEC and, therefore, require exotic matter. However, the theorems which guarantee this violation do not address how much exotic matter is needed in such spacetimes. As mentioned in section 2.3, some quantum effects are known to generate small energy condition violations (for a more detailed study of known energy condition violations, see chapter 12.3 of Visser, 1995). In other words, if wormhole solutions require sufficiently small amounts of exotic matter, energy requirements may become less of an obstacle to the existence of such geometries. To find out how much exotic matter is needed to sustain traversable wormhole geometries, Visser, Kar, and Dadhich quantified the amount of ANEC-violating matter by using the integral $\oint(\rho-\tau)dV$, similar to the simplest formula for total mass, $\oint\rho dV$. When using this integral for a wormhole whose field only deviates from Schwarzschild in the region from the throat out to some radius a, they found lower and upper bounds to the ANEC integral (see energy condition 6 in section 1). As a result, they concluded that it is possible to construct traversable wormholes with arbitrarily small quantities of ANEC-violating matter by a choice of suitable a and $\phi(r)$ (Visser, et al., 2003).

3.3 Wormholes and Black Holes

Hayward also made significant contributions to the definition of wormholes. His approach was to study the relationship between black hole and wormhole geometries. He provided a general definition of black holes in terms of their trapping horizons (types of hypersurfaces foliated by marginal surfaces) and derived general laws of black hole dynamics (Hayward, 1994). This prompted the discovery of a unified first law of black hole dynamics and relativistic thermodynamics in spherically symmetric general relativity (Hayward, 1998b), followed by further contributions to black hole thermodynamics (Hayward, 1998a).

He then realized wormholes and black holes are very similar: in terms of local properties, both are defined by the presence of marginally trapped surfaces and may be defined by outer trapping horizons. For a static black hole, the event horizon is an outer trapping horizon, and for a static wormhole, the wormhole throat is a double outer trapping horizon. The difference is in the causal nature of the trapping horizons. As such, black holes and wormholes can be locally defined by outer trapping horizons that are respectively achronal (space-like or null) and temporal (time- like) (Hayward, 1999). In this framework, Hayward (2009) derived the laws of wormhole dynamics in spherical symmetry, analogous to the laws of black hole dynamics, and suggested a new area of wormhole thermodynamics.

Using the Callan-Giddings-Harvey-Strominger (CGHS) 2D dilaton gravity model (Callan, et al., 1992), Hayward (2002) provided examples of construction of wormholes from black holes (by irradiating a static black hole with a massless ghost Klein-Gordon field, one with a negative gravitational coupling) (Koyama and Hayward, 2004), operation of wormholes for transport or signaling (by demonstrating that the wormhole stays traversable for a long time when a small enough pulse is sent through it), maintenance of an operating wormhole (by preceding the pulse sent to demonstrate operation with one of equal and opposite energy, returning the wormhole to its initial state), and collapse of a wormhole to a black hole (by "switching off" the supporting ghost radiation from both sides of the wormhole). Black holes and wormholes are thus interconvertible, which can be understood in terms of changes in the causal nature of the geometry's outer trapping horizon. If one assumes the supply of a ghost field to be reasonable, Hayward's model makes the possibility of wormholes as technological tools more tangible by providing specific mathematical means for the use of such solutions. His idea of creating wormholes from black holes is particularly useful, since black holes are better known objects which have been proven to exist.

3.4 Stability and Assembly

One important aspect of traversable wormhole solutions not covered in the previous sections is wormhole stability (requirement 6). This is because numerical simulations that allow for a study of stability require formalisms that separate the Einstein field equations, and the reader is not expected to be familiar with such approaches. Here, the results obtained from using these techniques are summarized with the purpose of situating the reader in the current research scenario.

The 2+2 approach to general relativity (in which the gravitational field is decomposed with respect to two intersecting foliations of null surfaces) has been used to simulate a numerically stable static EBMT wormhole. The simulated wormhole was, however, unstable to perturbations (Hayward, 1993). More recently, the 3+1 formalism (in which spacetime is split into three-dimensional space and time) was used to demonstrate the dynamic instability of EBMT wormholes (González, et al., 2008), and analytic work has been done to explore linear instability of dynamical wormholes (Bronnikov, et al., 2012). Overall, most results show that potentially traversable wormholes are unstable. This adds to the lack of plausibility of such solutions, previously pointed out due to their energetic requirements (section 2.3).

Wormhole assembly (requirement 7) was, like stability, unapproached in previous sections. This is because little is known about potential wormhole assembly mechanisms. To create a shortcut between two regions of space, the topology of spacetime would have to be altered, which is likely impossible. Therefore, one would need to find an existing wormhole and fit it to their needs. One way this could be accomplished is through the spacetime foam, the quantum fluctuation of spacetime on small scales due to quantum mechanics (Wheeler, 1955). Small wormholes could form and vanish due to these fluctuations in a fraction of a second. Advanced civilizations could somehow amplify one of them and stabilize it with exotic matter. It is worth noting that even if microscopic wormholes did form due to quantum fluctuations, physicists do not yet know any possible mechanisms for amplifying them. As mentioned in the previous section, mechanisms for creating wormholes from black holes have also been proposed. The problem then becomes, once again, the need for exotic matter.

3.5 Rotating Wormholes

Non-EBMT wormholes have also been proposed and studied, with a big focus on generalizing Morris and Thorne's work to stationary, rotating, axially symmetric wormholes. In 1998, Teo constructed the stationary and axially symmetric generalization of the EBMT wormhole by following the same method as Morris and Thorne did in their article: finding the most general metric with the desired symmetries and examining the conditions under which it would describe a traversable wormhole. The general metric for a stationary, axisymmetric spacetime is

$$ds^2 = -N^2dt^2 + e^u dr^2 + r^2K^2[d\theta^2 + sin^2\theta(d\varphi - \omega dt)^2] \quad (29)$$

and the general metric for a traversable wormhole with such symmetries, which he obtained after imposing the flare- out condition at the throat and traversability requirements 1 and 2, is

$$ds^2 = -N^2dt^2 + (1 - \frac{b}{r})^{-1}dr^2 + r^2K^2[d\theta^2 + sin^2\theta(d\varphi - \omega dt)^2] \quad (30)$$

where N, b, K and ω are functions of r and θ. Here, N is the analog of the redshift function ϕ in Equation 2 (in the limit of zero rotation and spherical symmetry, it reduces to $e^{\phi(r)}$). It must be finite and nonzero to avoid horizons and curvature singularities. As in the static case, b is the shape function which obeys $b \leq r$. At the throat, it must be independent of θ to avoid curvature singularities and satisfy the flare-out condition. It is important to note that by imposing the flare-out condition, Teo arrived at Equation 21, which, as it did in the static case, leads to the necessity of exotic matter. In the metric above, K determines the proper radial distance and ω controls the angular velocity of the wormhole.

Teo's work showed that stationary, axisymmetric wormholes violate the null energy condition. However, it did not consider what type of matter could generate this geometry. To address this, Bergliaffa and Hibberd (2000) showed that the metric in Equation 29 can only be generated by a fluid with a nonzero stress tensor if two additional conditions restricting the Einstein tensor are met. They also show that neither a perfect fluid nor a fluid with anisotropic stresses could generate the wormhole whose metric is given in Equation 30.

Finally, in 2004, Kuhfittig generalized Teo's solution to a time-dependent rotating wormhole by assuming ω to be a function of r, θ, and t. His results showed that the magnitude of the angular velocity may have little effect on the weak energy condition violations of axisymmetric wormholes. However, increasing angular velocity makes these violations less severe for spherically symmetric solutions. He also found that the violation of the WEC by time-dependent axially symmetric wormholes is much less severe than the violations required by the EBMT wormhole. The radial tidal constraint (Equation 24) becomes easier to meet due to the rotation.

In general, efforts to generalize the EBMT wormhole to rotating solutions have been motivated by practical arguments: if an advanced civilization were to construct a wormhole, it would most likely be rotating and changing with time. All solutions studied so far have been found to inevitably violate the energy conditions.

3.6 Wormholes in beyond-GR Theories

Wormholes have been studied in the context of different gravity theories, like R^m gravity and string theory. The reader is not expected to be familiar with such theories, but advances related to these wormholes are mentioned here for the completeness of this Recent Progress section. Bronnikov, Konoplya, and Zhidenko studied the instabilities of AdS wormholes (Bronnikov, et al., 2012). Recently, Maldacena and collaborators have been constructing traversable wormholes with string theory-motivated arguments. Specifically, they have analyzed wormholes in nearly-AdS_2 gravity (Maldacena, et al., 2017), studied the formation of SYK wormholes (Maldacena and Milekhin, 2020), proposed traversable wormholes within the Randall-Sundrum model (Maldacena and Milekhin, 2021), and proposed bra-ket wormholes (Chen, et al., 2021).

Traversable wormholes leading to causality violations are not allowed due to the achronal averaged null energy condition (a weaker version of the ANEC with no known violations) (Graham and Olum, 2007). However, sufficiently long wormholes (for which it takes longer to go through the wormhole than through the ambient space) could be supported by quantum effects. Maldacena, Milekhin, and Popov (Maldacena, et al., 2018) found an example of this by considering a solution which can be viewed as a pair of entangled black holes with an interaction term generated by the exchange of fermion fields responsible for generating a negative Casimir-like vacuum energy. The two black hole throats are joined to each other by a spacetime that looks like $AdS_2 \times S_2$, but in global coordinates. This solution is embedded in the Standard Model by making its overall size small compared to the electroweak scale. This discovery sparked new interest in wormholes within the theoretical physics community, since it is an example of a wormhole solution which causes no causality violations and is supported by a known quantum violation of the ANEC.

4. Conclusion

Despite the pedagogical value of wormhole geometries, the possibility of constructing them in the foreseeable future seems slim. Wormhole solutions found in general relativity have thus far been dynamically unstable and typically require large amounts of exotic matter to achieve static stability. Even though wormholes which either do not require exotic matter or can be supported by known energy condition violations have been proposed, they only exist in the context of beyond-GR theories of gravity.

The main reason to continue studying wormhole geometries is to better understand the limits of general relativity and energy conditions. Wormhole research has led to interesting developments in numerical relativity (Hayward, 1993), research on hyperfast travel (Krasnikov, 1998), and research on energy condition violations (Kar, et al., 2004), including the quantum inequalities (Ford and Roman, 1978).

In light of the mathematical connections between wormhole and black hole geometries, one might hope that wormholes could someday become as observationally justified as black holes. However, there is an important physical distinction to keep in mind. Black holes were predictions from general relativity resulting from natural stress-energy tensors, meaning that the conditions under which a black hole can spontaneously form are not hard to find in nature. On the other hand, traversable wormholes are "reverse-engineered" solutions: some requirements are listed which the desired solution must satisfy; a metric is found which satisfies those conditions; and the stress-energy required to form such a geometry is computed, leading to the realization that the required material has never been observed or predicted in sufficient amounts. Therefore, wormholes remain a theoretical exercise for the time being, except in the realm of science fiction.

While it is certainly possible for unexpected technological advances to change this current scenario, a theorist who is interested in predicting observable results might take a different approach to wormhole research. Equipped with the rigorous definitions of wormhole geometry discussed here, rather than attempting to construct a geometry obeying traversability requirements, one might instead ask the question: what is the most physically realistic scenario in which there is a chance of detecting a region of spacetime obeying the flare-out condition of a wormhole throat? This could help develop constraints on the energy scales of naturally occurring wormholes, if they exist. In other words, one might shift the focus from finding traversable wormhole solutions to investigating the most plausible wormhole solution and its potential observable consequences, even if it is non-traversable.

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References

Bergliaffa, S.E.P, & Hibberd, K.E. (2000). On the stress-energy tensor of a rotating wormhole. *arXiv e-prints,* gr- qc/0006041.

Bronnikov, K.A. (1973). Scalar-tensor theory and scalar charge. *Acta Phys. Pol., B*, v. B4, no. 2, 251-256.

Bronnikov, K.A., Konoplya, R.A., & Zhidenko, A. (2012). Instabilities of wormholes and regular black holes supported by a phantom scalar field. *Physical Review D,* 86. 2, 024028. https://doi.org/10.1103/physrevd.86.024028.

Callan, C.G., Giddings, S.B, Harvey, J.A., & Strominger, A. (1992). Evanescent black holes. *Physical Review D*, 45. 4, R1005-R1009. https://doi.org/10.1103/PhysRevD.45.R1005.

Carroll, S.M. (2003). *Spacetime and Geometry: An Introduction to General Relativity.* Addison-Wesley.

Chen, Y., Gorbenko, V., & Maldacena, J. (2021). Bra-ket wormholes in gravitationally prepared states. *Journal of High Energy Physics,* 2021. 2, article 9. https://doi.org/10.1007/jhep02(2021)009.

Einstein, A., & Rosen, N. (1935). The Particle Problem in the General Theory of Relativity. *Physical Review,* 48. 1, 73-77. https://doi.org/10.1103/PhysRev.48.73.

Ellis, H.G. (1973). Ether flow through a drainhole: A particle model in general relativity. *Journal of Mathematical Physics,* 14. 1, 104-118. https://doi.org/10.1063/1.1666161.

Ford, L.H., & Roman, T.A. (1978). Averaged Energy Conditions and Quantum Inequalities. *Physical Review D,* 51. 8, 4277-4286. http://dx.doi.org/10.1103/PhysRevD.51.4277.

Friedman, J.L., Schleich, K., & Witt, D.M. (1993). Topological Censorship. *Physical Review Letters,* 71. 10, 1486- 1489. https://doi.org/10.1103/PhysRevLett.71.1486.

Fronsdal, C. (1959). Completion and Embedding of the Schwarzschild Solution. *Phys. Rev.,* 116, 778-781. https://doi.org/10.1103/PhysRev.116.778.

González, J.A., Guzmán, F.S., & Sarbach, O. (2008). Instability of wormholes supported by a ghost scalar field: II. Nonlinear evolution. *Classical and Quantum Gravity,* 26. 1, 015011. https://doi.org/10.1088/0264-9381/26/1/015011.

Graham, N., & Olum, K.D. (2007). Achronal averaged null energy condition. *Physical Review D,* 76, 064001. https://doi.org/10.1103/PhysRevD.76.064001.

Hawking, S., & Ellis, G. (1973). *The Large Scale Structure of Space-Time* (Cambridge Monographs on Mathematical Physics). Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9780511524646.

Hayward, S.A. (1993). Dual-null dynamics of the Einstein field. *Classical and Quantum Gravity,* 10. 4, 779-790. https://doi.org/10.1088/0264-9381/10/4/013.

Hayward, S.A. (1994). General laws of black-hole dynamics. *Physical Review D,* 49. 12, 6467-6474. https://doi.org/10.1103/PhysRevD.49.6467.

Hayward, S.A. (1998a). Relativistic thermodynamics. *arXiv e-prints*, gr-qc/9803007.

Hayward, S.A. (1998b). Unified first law of black-hole dynamics and relativistic thermodynamics. *Classical and Quantum Gravity*, 15. 10, 3147-3162. https://doi.org/10.1088/0264-9381/15/10/017.

Hayward, S.A. (1999). Dynamic wormholes. *International Journal of Modern Physics D,* 08. 03, 373-382. https://doi.org/10.1142/s0218271899000286.

Hayward, S.A. (2002). Black holes and traversable wormholes: a synthesis. *arXiv e-prints,* gr-qc/0203051.

Hayward, S.A. (2009). Wormhole dynamics in spherical symmetry. *Physical Review D*, 79. 12, 124001. https://doi.org/10.1103/PhysRevD.79.124001.

Hochberg, D., & Visser, M. (1997). Geometric structure of the generic static traversable wormhole throat. *Physical Review D,* 56.8, 4745-4755. https://doi.org/10.1103/physrevd.56.4745.

Hochberg, D., & Visser, M. (1998). Dynamic wormholes, antitrapped surfaces, and energy conditions. *Physical Review D,* 58. 4, 044021. https://doi.org/10.1103/PhysRevD.58.044021.

Kar, S., Dadhich, N., & Visser, M. (2004). Quantifying energy condition violations in traversable wormholes. *Pramana,* 63. 4, pp. 859-864. http://dx.doi.org/10.1007/BF02705207.

Krasnikov, S.V. (1998). Hyperfast travel in general relativity. *Physical Review D,* 57. 8, pp. 4760-4766. http://dx.doi.org/10.1103/PhysRevD.57.4760.

Kruskal, M.D. (1960). Maximal Extension of Schwarzschild Metric. *Phys. Rev.,* 119, 1743-1745. https://doi.org/10.1103/PhysRev.119.1743.

Koyama, H., & Hayward, S.A. (2004). Construction and enlargement of traversable wormholes from Schwarzschild black holes. *Physical Review D,* 70.8, 084001. https://doi.org/10.1103/PhysRevD.70.084001.

Kuhfittig, P.K.F. (2003). Axially symmetric rotating traversable wormholes. *Physical Review D,* 67. 6, 064015. https://doi.org/10.1103/physrevd.67.064015.

Maldacena, J., & Milekhin, A. (2020). SYK wormhole formation in real time. *Journal of High Energy Physics,* 2021.4, article 258. https://doi.org/10.1007/jhep04(2021)258.

Maldacena, J., & Milekhin, A. (2021). Humanly traversable wormholes. *Physical Review D,* 103.6, 066007. https://doi.org/10.1103/physrevd.103.066007.

Maldacena, J., Milekhin, A., & Popov, F. (2018). Traversable wormholes in four dimensions. *arXiv e-prints,* arXiv:1807.04726.

Maldacena, J., Stanford, D., & Yang, Z. (2017). Diving into traversable wormholes. *Fortschritte der Physik,* 65. 5, 1700034. https://doi.org/10.1002/prop.201700034.

Morris, M.S., & Thorne, K.S. (1988a). Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity. *American Journal of Physics,* 56. 5, 395-412. https://doi.org/10.1119/1.15620.

Morris, M.S., & Thorne, K.S. (1988b). Wormholes, Time Machines, and the Weak Energy Condition. *Phys. Rev. Lett.,* 61, 1446-1449. https://doi.org/10.1103/PhysRevLett.61.1446.

Schwarzschild, K. (1916). On the Gravitational Field of a Mass Point According to Einstein's Theory. *Abh. Konigl. Preuss. Akad. Wissenschaften Jahre,* 1916, 189-196.

Teo, E. (1998). Rotating traversable wormholes. *Physical Review D,* 58. 2, 024014. https://doi.org/10.1103/PhysRevD.58.024014.

Vishveshwara, C.V. (1968). Generalization of the "Schwarzschild Surface" to Arbitrary Static and Stationary Metrics. *Journal of Mathematical Physics,* 9. 8, 1319-1322. https://doi.org/10.1063/1.1664717.

Visser, M. (1995). *Lorentzian wormholes: from Einstein to Hawking*. AIP Press.

Visser, M., Kar, S., & Dadhich, N. (2003). Traversable Wormholes with Arbitrarily Small Energy Condition Violations. *Physical Review Letters,* 90.20, 201102. https://doi.org/10.1103/physrevlett.90.201102.

Wheeler, J.A. (1955). Geons. *Physical Review,* 97.2, 511-536. https://doi.org/10.1103/PhysRev.97.511.